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RESEARCH NOTE

ON THE RELATION BETWEEN SUMMATION AND FACILITATION

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Abstract—*Summation* refers to the small reduction of threshold contrast as a signal is extended in space or time, or when multiple signals are presented simultaneously. *Facilitation* refers to the large reduction in the threshold contrast increment when it is superimposed on a near-threshold pedestal contrast. Separate hypotheses relate each of these effects to distinct measures of the log–log steepness of the psychometric function. This note shows that for 2-alternative forced-choice (2afc) proportion correct, the Weibull function $P(c) = 1\text{-}(1\text{-}\gamma)\exp[-(c/\alpha)^\beta]$ and the d' power law $d'(c) = (c/c_{0.76})^b$ are practically indistinguishable. Furthermore, their steepness parameters are proportional to each other, $\beta = 1.247\ b$. This implies a simple relation between 2afc summation and facilitation which can be tested without measuring the psychometric function.

Probability summation Facilitation Contrast discrimination Psychometric function

INTRODUCTION

Until recently, summation and facilitation have been treated as independent, unrelated phenomena. However, the generally accepted models of the two phenomena imply a connection, via the psychometric function. This connection was obscured by the use of different descriptions of the psychometric function in the two models. It is shown here that the two different descriptions of the psychometric function are practically indistinguishable, and equations are derived relating the parameters of the two descriptions.

A quantitative relationship between summation and facilitation has already been deduced for a particular model of visual detection by Pelli (1985). There it was argued that the popular assumptions of probability summation and that the observer uses a decision variable together imply that the observer acts as though uncertain of the physical parameters of the signal. A simple model of visual detection incorporating uncertainty explained not only summation and facilitation, but many other aspects of contrast detection and discrimination as well. The 2afc performance of that uncertainty model turned out to be characterized primarily by a single parameter, the degree of uncertainty. This parameter could be estimated from the steepness of the psychometric function, the amount of summation, or the amount of facilitation. As a result any one of these effects was sufficient to predict the others. I will show here that this property is not unique to that model of detection, and in fact follows directly from the popular explanations for summation and facilitation.

The psychometric function

For "probability summation" calculations the human psychometric function $P(c)$, describing the probability of a correct response as a function of contrast c, is usually fit by a *Weibull function*

$$W(c) = 1 - (1-\gamma)\exp[-(c/\alpha)^\beta] \qquad (1)$$

where c is signal contrast and α, β and γ are parameters (e.g. Quick, 1974; Graham, 1977). α is the threshold contrast, at which $P(\alpha) = W(\alpha) = 1 - (1-\gamma)/e$; β determines the log–log steepness; and γ is the "guessing" rate $W(0) = \gamma$. For 2afc experiments γ is 0.5, so the threshold criterion $P(\alpha)$ is 0.816.

For "d′ additivity" calculations the human psychometric function is usually fit by a d' *power law.*

$$d' = (c/c_{0.76})^b \qquad (2)$$

where $c_{0.76}$ is the contrast at which d' is 1 (and the proportion correct is 0.76), and b is the log–log steepness (Nachmias and Sansbury, 1974; Foley and Legge, 1981; Pelli, 1985), d' is defined by

$$d' = \sqrt{2}\, N^{\text{inv}}[P(c)] \tag{3}$$

where $P(c)$ is the 2afc proportion correct at contrast c, and $N^{\text{inv}}(\ )$ is the functional inverse of the cumulative normal distribution $N(\ )$, i.e. $x = N^{\text{inv}}[N(x)]$. The cumulative normal distribution $N(x)$ is the probability that a sample from a unit-variance Gaussian distribution will be less than x.

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp[-t^2/2]\,\mathrm{d}t. \tag{4}$$

The relation between equations (1) and (2) is explored in the Analysis Section.

Summation

The hypothesis of *probability summation* boils down to two equations

$$\frac{1 - P_K(c)}{1 - P_K(0)} = \left(\frac{1 - P_1(c)}{1 - P_1(0)}\right)^K \tag{5}$$

and

$$P_K(0) = P_1(0) \tag{6}$$

where $P_1(c)$ is the proportion correct at contrast c when only one mechanism is stimulated by the signal, and $P_K(c)$ is the proportion correct when K mechanisms are identically stimulated by the signal (Chaplin, 1880; Weibull, 1951; Brindley, 1954; Quick, 1974; Green and Luce, 1975; Graham, 1977). This hypothesis can be tested without any assumptions provided K is known (to within a proportionality constant) for at least two experimental conditions (e.g. Kramer *et al.*, 1985).

If either $P_K(c)$ or $P_1(c)$ is a Weibull function then both are Weibull functions, and their parameters are related in a simple way: $\beta_K = \beta_1$, $\gamma_K = \gamma_1$, and

$$\alpha_K = K^{-1/\beta_1}\alpha_1 \tag{7}$$

where α_K, β_K, and γ_K are the parameters of the Weibull function describing $P_K(c)$ and α_1, β_1, and γ_1 are the parameters of the Weibull function describing $P_1(c)$. Equation (7) tells us that the graph of log α_K vs log K has a slope of $-1/\beta_1$. For stimuli to which the observer is equally sensitive and which are thought to be detected by independent mechanisms, it is generally assumed that K equals the number of stimuli. Often it is assumed that K is proportional to the signal duration or spatial extent. Legge (1978) reported log–log slopes, s, of threshold versus duration with an overall average of -0.29 ± 0.08 (the "$\pm$" specifies a 95% confidence interval) for detection of sinusoidal gratings presented for 180–1000 msec at spatial frequencies of 1.5–12 c/deg. Robson and Graham (1981) reported a log–log slope, s, of threshold vs number of bars of -0.29 for gratings of various spatial frequencies presented in a region of uniform sensitivity. We may summarize these estimates of the log–log slope for summation by the corresponding value of $\beta = -1/s = 3.5 \pm 0.8$.

Facilitation

d' additivity is the hypothesis that the d' for discriminating two contrasts c_1 and c_2 will equal the difference in d' values for detecting the two contrasts.

$$d'_{1,2} = d'_2 - d'_1 \tag{8}$$

Note that d' is defined by equation (3) to be a simple transformation of the proportion correct. Thus the d' additivity hypothesis refers only to measurable quantities, and can be tested directly, without any assumptions (Nachmias and Sansbury, 1974; Foley and Legge, 1981).

Substituting for d'_1 and d'_2 from the d' power law [equation (2)] and rearranging yields

$$c_2 - c_1 = (d'_{1,2}\, c^b_{0.76} + c^b_1)^{1/b} - c_1. \tag{9}$$

Substituting for $d'_{1,2}$ from the definition of d' [equation (3)] and switching to the usual notation for contrast discrimination experiments, obtain

$$\Delta c_p = (\sqrt{2}\, N^{\text{inv}}(p)\, c^b_{0.76} + c^b)^{1/b} - c \tag{10}$$

where $\Delta c_p = c_2 - c_1$ is the discrimination-threshold contrast increment for proportion correct p and pedestal contrast $c = c_1$. When the pedestal c is zero, the discrimination threshold Δc_p is equal to the detection threshold c_p. Substituting, $c^b_p = \sqrt{2}\, N^{\text{inv}}(p)\, c^b_{0.76}$, yields

$$\Delta c_p = (c^b_p + c^b)^{1/b} - c \tag{11}$$

where $c_p = [\sqrt{2}\, N^{\text{inv}}(p)\, c^b_{0.76}]^{1/b}$ is the detection-threshold contrast for proportion correct p. Equation (11) predicts the increment threshold Δc_p at all pedestals c from knowledge of just the detection threshold c_p, and the psychometric steepness b.

The prediction is valid only at contrasts at which the d' power law [equation (2)] is valid, i.e. sub- and near-threshold contrasts. There is evidence that d' tends to saturate at supra-threshold contrasts, which would account for the eventual rise of Δc_p with supra-threshold contrast c (Nachmias and Sansbury, 1974;

Legge and Foley, 1980; Wilson, 1980; Legge, 1981; Legge and Kersten, 1983; Thomas, 1983).

If the pedestal is at threshold, i.e. $c = c_p$, then we can rearrange and simplify equation (11) to obtain

$$\Delta c_p/c_p = 2^{1/b} - 1. \tag{12}$$

Facilitation may be summarized by the ratio $\Delta c_p/c_p$. Equation (12) predicts that this ratio will be independent of the threshold criterion p. Six estimates of this ratio from the data of Foley and Legge (1981) average 0.31 ± 0.10 and two estimates from data of Nachmias and Sansbury (1974) average 0.32 ± 0.14. Taking $\Delta c_p/c_p = 0.31 \pm 0.10$ and solving equation (12) for b we obtain $b = 1/\log_2 (1 + \Delta c_p/c_p) = 2.6 \pm 0.7$. The Analysis Section will relate the β of summation to the b of facilitation, allowing us to compare the results of the two kinds of experiment.

ANALYSIS

To show that the Weibull [equation (1)] and d' power law [equation (2)] fits are practically indistinguishable let us first fit the cumulative normal by the Weibull function. What values of α, β, and γ minimize the maximum error $|W(x) - N(x)|$ for all positive x? (I minimize the *maximum* error rather than some average statistic such as *mean square* error because a particular application may depend on only a few points of the psychometric function.) A computer program (Chandler, 1965) was used to find these "minimax" values of the parameters α, β and γ which minimize ϵ, where ϵ is the maximum error for all positive x

$$\epsilon(\alpha, \beta, \gamma) = \max_{x \geq 0} |W(x) - N(x)|. \tag{13}$$

This required finding ϵ, the maximum error, for each test combination of α, β, and γ. The cumulative normal was computed by means of the approximations of Abramowitz and Stegun (1964).

The result appears in the first row of Table 1. The second row shows the minimax values of α and β if γ is constrained to be 0.5, which is appropriate for 2afc experiments.

Table 1. Parameters of the Weibull function $W(x)$ to best fit the cumulative normal function $N(x)$*

	α_ϵ	β_ϵ	γ_ϵ	ϵ
Minimum ϵ	0.886	1.283	0.5054	0.0062
Minimum ϵ, given $\gamma = 0.5$	0.875	1.247	0.5	0.0076

*ϵ is the maximum error, defined in equation (13), rounded up.

We can now show that the psychometric functions underlying the d' power law and the Weibull function are practically indistinguishable. First we need to re-express the d' power law as proportion correct. Using equation (3) to substitute for d' in equation (2), and solving for P yields

$$P(c) = N[(c/c_{0.76})^b/\sqrt{2}]. \tag{14}$$

Let us take equation (14) as the definition of $P(c)$. Taking x in equation (13) to be $(c/c_{0.76})^b/\sqrt{2}$ we may substitute $W(x)$ for $N(x)$ in equation (14) with an error of at most $\pm\epsilon$

$$P(c) = W[(c/c_{0.76})^b/\sqrt{2}] \pm \epsilon \tag{15}$$

where α_ϵ, β_ϵ, γ_ϵ, and ϵ are taken from the second row of Table 1. Using equation (1) to expand the Weibull function in equation (15) yields

$$P(c) = 1 - (1 - \gamma_\epsilon) \times \exp[\![-[\{(c/c_{0.76})^b/\sqrt{2}\}/\alpha_\epsilon]^{\beta_\epsilon}]\!] \pm \epsilon \tag{16}$$

which readily simplifies to

$$P(c) = 1 - (1 - \gamma)\exp[-(c/\alpha)^\beta] \pm \epsilon \tag{17}$$

where

$$\alpha = (\sqrt{2}\,\alpha_\epsilon)^{1/b}\, c_{0.76} = 1.237^{1/b}\, c_{0.76} \tag{18}$$

$$\beta = \beta_\epsilon b = 1.247 b \tag{19}$$

$$\gamma = \gamma_\epsilon = 0.5. \tag{20}$$

A glance at the definition of the Weibull function in equation (1) shows that our desired result follows immediately from equation (17)

$$P(c) = W(c) \pm \epsilon. \tag{21}$$

This states that whenever we know the parameters $c_{0.76}$ and b of a fit by a d' power law [equation (2)], we can use equations (18)–(20) to compute the parameters α, β and γ of a Weibull fit [equation (1)] and vice versa. The two fits will differ by no more than ϵ.

Is ϵ measurable? For 2afc the Weibull and d'-power-law fits differ by no more than $\epsilon = 0.0076$. If $P < 0.99$ then this difference is too small to resolve with even 1000 trials. If $P > 0.99$, then the difference would be significant after 1000 trials, but it is doubtful that the subject's probability of response is stationary to a fraction of a percent. Thus the two fits are practically indistinguishable.

DISCUSSION

The purpose of this note is to point out a shortcut. Rather than fitting a Weibull function to the psychometric data to predict summation, and then fitting a d' power law to predict facilitation, one can compute the parameters of one fit directly from the other, or even predict facilitation directly from the measured summation, or vice versa.

The key result is equation (19), which relates the β of summation [equation (7)] to the b of facilitation [equation (12)]. Pelli (1985) reported an equation (5.6) very similar to equation (19), as an empirical result of fitting Weibull and d' power law functions to 2afc psychometric functions of an "uncertainty" model. In the Introduction it was noted that facilitation data yield $b = 2.6 \pm 0.7$. Now, by equation (19), $\beta = 1.247b$, so β should be 3.3 ± 0.9 for facilitation, which is in good agreement with the estimate of β from summation data of 3.5 ± 0.8.

It was concluded that we cannot measure the psychometric function sufficiently precisely to detect the difference ϵ. However, even though measuring the psychometric function will not distinguish the two theoretical psychometric functions (Weibull and d' power law) it may be that the two theoretical functions do make distinguishable predictions when combined with another hypothesis. For example, with the probability summation hypothesis, the d' power law would predict an effect of summation on the psychometric steepness, b or β, while the Weibull function predicts no effect. The difference in these predictions may be measurable because the predictions tend to amplify small differences in the psychometric function at low sub-threshold contrasts. Thus, by using ancillary hypotheses, it may be possible to distinguish the two functions, but they will still be useful approximations over some range of near-threshold contrasts.

Comparison with Wilson's (1980) *result*

It is shown above that psychometric data which are well described by a Weibulll function with parameters, α, β, and $\gamma = 0.5$, will also be well described by a d' power law with parameters $c_{0.76}$ and b given by equations (18) and (19).

$$d' = 1.237\,(c/\alpha)^{\beta/1.247} \tag{22}$$

A more direct approach to calculating d' from a Weibull function is to substitute $W(c)$ from equation (1) for $P(c)$ in equation (3), yielding

$$d' = \sqrt{2}\,N^{\text{inv}}[1-(1-\gamma)\,e^{-(c/\alpha)^{\beta}}]. \tag{23}$$

However, this is impractical due to the lack of an analytic expression for the inverse cumulative normal $N^{\text{inv}}(\ )$. Wilson (1980) set $\gamma = 0.5$ and used an analytic approximation for $N(\ )$, which could be inverted to obtain an approximation for $N^{\text{inv}}(\ )$, to obtain a usable formula

$$d' \approx \sqrt{2}\,N^{\text{inv}}(0.75) \times \{[1+(\ln 2)\,(c/\alpha)^{\beta}]^{1/3}-1\}/(2^{1/3}-1) \tag{24}$$

where I have made the following substitutions for Wilson's notation: "σ" $= 1/N^{\text{inv}}(0.75)$, "F" $= d'/[\sqrt{2}\,N^{\text{inv}}(0.75)]$, "$C$" $= c$, "S" $= (\ln 2)^{1/\beta}/\alpha$, "$Q$" $= \beta$, and "k" $= 2^{1/3}-1$. Equation (24) may be a more accurate way to calculate d' from the Weibull function than is equation (22), but this will be irrelevant when the precision of actual data is considered. For most analyses, such as d' additivity, the simple d' power law [equation (2)] is preferable to the unwieldy expression in equation (24).

CONCLUSION

Both the d' power law and the Weibull function have been widely used to fit and model 2afc psychometric data. It was shown here that these two functions are practically indistinguishable. The parameters of the two functions are related by equations (18)–(20).

Since predictions of summation [equation (7): $\alpha_K = K^{-1/\beta}\,\alpha_1$] and facilitation [equation (11): $\Delta c_p/c_p = 2^{1/b} - 1$] depend primarily on the log–log steepness of the psychometric function, it follows that the two predictions must be related [equation (19): $\beta = 1.247\,b$].

Although the two psychometric functions are very similar, they may not be equivalent for all purposes. It is possible that in combination with another hypothesis these two functions may yield distinguishable predictions.

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